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Problem 1

a) For $l=1$, $|\vec{L}| = \hbar\sqrt{l(l+1)} = \sqrt{2}\,\hbar$

b) The eigenvalues can be calculated with the eigenvalue equations for $\hat{L}_x$

For $|+_x\rangle$, $\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}\frac{1}{2}\\\frac{1}{\sqrt{2}}\\\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\1\\\frac{1}{\sqrt{2}}\end{pmatrix}\cdot\frac{\hbar}{\sqrt{2}} = +\hbar\begin{pmatrix}\frac{1}{2}\\\frac{1}{\sqrt{2}}\\\frac{1}{2}\end{pmatrix} \Rightarrow$ its eigenvalue is $\underline{\underline{+\hbar}}$

For $|0_x\rangle$, $\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{pmatrix} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}0\\0\\0\end{pmatrix} = 0\hbar\begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\-\frac{1}{\sqrt{2}}\end{pmatrix} \Rightarrow$ its eigenvalue is $\underline{\underline{0\hbar}}$ ($\hbar$ for unit)

For $|-_x\rangle$, $\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}\frac{1}{2}\\-\frac{1}{\sqrt{2}}\\\frac{1}{2}\end{pmatrix} = \frac{\hbar}{\sqrt{2}}\begin{pmatrix}-\frac{1}{\sqrt{2}}\\1\\-\frac{1}{\sqrt{2}}\end{pmatrix} = -\hbar\begin{pmatrix}\frac{1}{2}\\-\frac{1}{\sqrt{2}}\\\frac{1}{2}\end{pmatrix} \Rightarrow$ its eigenvalue is $\underline{\underline{-\hbar}}$

c) $\langle\hat{L}_z\rangle = \langle\psi_1|\hat{L}_z|\psi_1\rangle$

$= \left(\sqrt{\tfrac{1}{8}}\ \sqrt{\tfrac{3}{8}}\ \sqrt{\tfrac{4}{8}}\right)\hbar\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}\begin{pmatrix}\sqrt{\frac{1}{8}}\\\sqrt{\frac{3}{8}}\\\sqrt{\frac{4}{8}}\end{pmatrix} = \left(\sqrt{\tfrac{1}{8}}\ \sqrt{\tfrac{3}{8}}\ \sqrt{\tfrac{4}{8}}\right)\begin{pmatrix}\sqrt{\frac{1}{8}}\\0\\-\sqrt{\frac{4}{8}}\end{pmatrix}\hbar$

$= \frac{1}{8}\hbar - \frac{4}{8}\hbar = -\frac{3}{8}\hbar$

$\langle\hat{L}_x\rangle = \langle\psi_1|\hat{L}_x|\psi_1\rangle$

$= \left(\sqrt{\tfrac{1}{8}}\ \sqrt{\tfrac{3}{8}}\ \sqrt{\tfrac{4}{8}}\right)\frac{\hbar}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}\sqrt{\frac{1}{8}}\\\sqrt{\frac{3}{8}}\\\sqrt{\frac{4}{8}}\end{pmatrix} = \left(\sqrt{\tfrac{1}{8}}\ \sqrt{\tfrac{3}{8}}\ \sqrt{\tfrac{4}{8}}\right)\begin{pmatrix}\sqrt{\frac{3}{8}}\\\sqrt{\frac{1}{8}}+\sqrt{\frac{4}{8}}\\\sqrt{\frac{3}{8}}\end{pmatrix}\frac{\hbar}{\sqrt{2}}$

$= \left(\frac{\sqrt{3}}{8} + \frac{\sqrt{3}}{8} + \frac{\sqrt{12}}{8} + \frac{\sqrt{12}}{8}\right)\cdot\frac{\hbar}{\sqrt{2}} = \frac{3}{8}\sqrt{6}\,\hbar$

d) You measure L_x for $l=1$. From problem 1b or general theory for L_x for $l=1$, the possible measurement results are the eigenvalues of $\hat{L}_x$ for $l=1$, which are $+\hbar$, $0\hbar$ and $-\hbar$.

$P_{+\hbar} = |\langle +_x|\psi_2\rangle|^2 = \left|\left(\tfrac{1}{2}\ \tfrac{1}{\sqrt{2}}\ \tfrac{1}{2}\right)\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\frac{1}{\sqrt{3}}\right|^2$

$= \left|\left(\frac{1}{2} + \frac{1}{\sqrt{2}} + \frac{1}{2}\right)\cdot\frac{1}{\sqrt{3}}\right|^2 = \left(\frac{2}{\sqrt{12}} + \frac{1}{\sqrt{6}}\right)^2 = \left(\frac{2+\sqrt{2}}{\sqrt{12}}\right)^2 = \frac{(2+\sqrt{2})^2}{12} \approx 0.97$

$P_{0\hbar} = |\langle 0_x|\psi_2\rangle|^2 = \left|\left(\tfrac{1}{\sqrt{2}}\ 0\ \tfrac{-1}{\sqrt{2}}\right)\begin{pmatrix}1\\1\\1\end{pmatrix}\frac{1}{\sqrt{3}}\right|^2$

$= \left|\left(\frac{1}{\sqrt{2}} + 0 - \frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{3}}\right|^2 = 0 \qquad = 0$

$P_{-\hbar} = |\langle -_x|\psi_2\rangle|^2 = \left|\left(\tfrac{1}{2}\ -\tfrac{1}{\sqrt{2}}\ \tfrac{1}{2}\right)\begin{pmatrix}1\\1\\1\end{pmatrix}\frac{1}{\sqrt{3}}\right|^2$

$= \left|\left(\frac{1}{2} - \frac{1}{\sqrt{2}} + \frac{1}{2}\right)\frac{1}{\sqrt{3}}\right|^2 = \left(\frac{2}{\sqrt{12}} - \frac{1}{\sqrt{6}}\right)^2 = \frac{(2-\sqrt{2})^2}{12} \approx 0.03$

Total probability $\quad 1$ +

Note

Given $P_{0\hbar} = 0$, the measurement outcome $L_x = 0\hbar$ will not occur for this state.

e) You can only properly calculate $\langle \hat{L}_z \rangle = \langle \psi | \hat{L}_z | \psi \rangle$ if you use $|\psi\rangle$ in a normalized form.

$$\langle \psi_3 | \psi_3 \rangle = \begin{pmatrix} -i & 2 & +i \end{pmatrix} \begin{pmatrix} +i \\ 2 \\ -i \end{pmatrix} = 1 + 4 + 1 = 6 \Rightarrow$$

To normalize $|\psi_3\rangle$ you must divide all probability amplitudes by $\sqrt{6}$ $\Rightarrow$

$$|\psi_3\rangle_N = \frac{i}{\sqrt{6}} |+_z\rangle + \frac{2}{\sqrt{6}} |0_z\rangle + \frac{-i}{\sqrt{6}} |-_z\rangle$$

$$\langle \hat{L}_z \rangle = {}_N\langle \psi_3 | \hat{L}_z | \psi_3 \rangle_N =$$

$$\hbar \begin{pmatrix} \frac{-i}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{+i}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} +\frac{i}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ -\frac{i}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} \frac{-i}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{+i}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} +\frac{i}{\sqrt{6}} \\ 0 \\ +\frac{i}{\sqrt{6}} \end{pmatrix} \hbar$$

$$= \left(\frac{-i}{\sqrt{6}} \frac{i}{\sqrt{6}} + 0 + \frac{-1}{6} \right) \hbar = \hbar \left(\frac{+1}{6} - \frac{1}{6} \right) = 0 \, \hbar$$

f) $$\Delta L_z = \sqrt{\langle \hat{L}_z^2 \rangle - \langle \hat{L}_z \rangle^2}$$

So, we must first calculate the matrix representation of $\hat{L}_z^2$

$$(\hat{L}_z)^2 \leftrightarrow \begin{pmatrix} +1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} +1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \hbar^2 = \begin{pmatrix} +1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & +1 \end{pmatrix} \hbar^2$$

Then

$$\langle \hat{L}_z^2 \rangle = \langle \psi_4 | \hat{L}_z^2 | \psi_4 \rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \hbar^2 =$$

$$\left(\frac{1}{2} + \frac{1}{2} \right) \hbar^2 = \hbar^2$$

$$\langle \hat{L}_z \rangle = \langle \psi_4 | \hat{L}_z | \psi_4 \rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = \left(\frac{1}{2} - \frac{1}{2} \right) \hbar = 0 \, \hbar$$

$\Rightarrow$

$$\Delta L_z = \sqrt{\langle \hat{L}_z^2 \rangle - \langle \hat{L}_z \rangle^2} = \sqrt{\hbar^2 - (0\hbar)^2} = \hbar$$

Problem 2

a) Ground state $\varphi_g(x) = \sqrt{\frac{2}{a}} \cos\left(\frac{\pi x}{a}\right)$

The energy is $E_n = \frac{n^2 \pi^2 \hbar^2}{2 m a^2}$ for $n=1 \Rightarrow$

$$E_g = \frac{\pi^2 \hbar^2}{2 m a^2}$$

b) Both particles can be in the ground state $\varphi_g(x)$. Label them 1 and 2, and write the two-particle state in symmetric form:

$$\varphi_S(x_1, x_2) = \frac{1}{\sqrt{2}}\left(\varphi_g(x_1)\varphi_g(x_2) + \varphi_g(x_2)\varphi_g(x_1)\right)$$

Is in this case not normalized since

$\varphi_g(x_1)\varphi_g(x_2) = \varphi_g(x_2)\varphi_g(x_1) \Rightarrow$

$$\varphi_S(x_1, x_2) = \frac{2}{a}\cos\left(\frac{\pi x_1}{a}\right)\cos\left(\frac{\pi x_2}{a}\right)$$

The energy gives E_g (see a)) for each particle, so the energy is $2E_g$

c) Both particles cannot be in the ground state, since this gives $\varphi_A(x_1, x_2) = 0$ (Pauli exclusion principle). So, the second particle must be in the state

$\varphi_e(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi x}{a}\right)$ with energy $E_e = 4E_g$ (see a))

Write down the two-particle state in antisymmetric form:

$$\varphi_A(x_1, x_2) = \frac{1}{\sqrt{2}}\left(\varphi_g(x_1)\varphi_e(x_2) - \varphi_g(x_2)\varphi_e(x_1)\right)$$

$$= \frac{\sqrt{2}}{a}\left(\cos\left(\frac{\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) - \cos\left(\frac{\pi x_2}{a}\right)\sin\left(\frac{2\pi x_1}{a}\right)\right)$$

with energy $E_g + E_e = 5E_g$ (as defined in a))

d) The probability for outcomes with such a measurement are governed by the probability density

$$|\varphi_A(x_1, x_2)|^2 = \varphi_A^*(x_1, x_2)\varphi_A(x_1, x_2)$$

$$= \frac{2}{a^2}\left(\cos^2\left(\frac{\pi x_1}{a}\right)\sin^2\left(\frac{2\pi x_2}{a}\right) - 2\cos\left(\frac{\pi x_1}{a}\right)\cos\left(\frac{\pi x_2}{a}\right)\sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) + \cos^2\left(\frac{\pi x_2}{a}\right)\sin^2\left(\frac{2\pi x_1}{a}\right)\right)$$

When exchanging the two particles, the probability density becomes

$$|\varphi_A(x_2, x_1)|^2 = \varphi_A^*(x_2, x_1)\varphi_A(x_2, x_1)$$

$$= \frac{2}{a^2}\left(\cos^2\left(\frac{\pi x_2}{a}\right)\sin^2\left(\frac{2\pi x_1}{a}\right) - 2\cos\left(\frac{\pi x_1}{a}\right)\cos\left(\frac{\pi x_2}{a}\right)\sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right) + \cos^2\left(\frac{\pi x_1}{a}\right)\sin^2\left(\frac{2\pi x_2}{a}\right)\right)$$

$$= |\varphi_A(x_1, x_2)|^2 \Rightarrow$$

The probabilities do not change.

Problem 3

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a) Short answer:

$$\frac{d\langle\hat{D}\rangle}{dt} = \frac{i}{\hbar}\langle\psi|[\hat{H},\hat{D}]|\psi\rangle = 0$$

since $[\hat{H},\hat{D}]=0$, $|\psi\rangle$ some arbitrary state

longer answer:

Say $|\psi_0\rangle = \alpha|\varphi_1\rangle + \beta|\varphi_2\rangle$

$|\psi(t)\rangle = e^{-\frac{iHt}{\hbar}}|\psi_0\rangle = \hat{U}|\psi_0\rangle \Rightarrow$

$$\langle\hat{D}(t)\rangle = \langle\psi(t)|\hat{D}|\psi(t)\rangle$$

$$= \langle\psi_0|\hat{U}^+\hat{D}\hat{U}|\psi_0\rangle$$

Use $\omega_1 = \frac{E_1}{\hbar}$

$\omega_2 = \frac{E_2}{\hbar}$

$$= \left(\alpha^* e^{+i\omega_1 t}\langle\varphi_1| + \beta^* e^{+i\omega_2 t}\langle\varphi_2|\right)\hat{D}\left(\alpha e^{-i\omega_1 t}|\varphi_1\rangle + \beta e^{-i\omega_2 t}|\varphi_2\rangle\right)$$

$$= |\alpha|^2\langle\varphi_1|\hat{D}|\varphi_1\rangle + |\beta|^2\langle\varphi_2|\hat{D}|\varphi_2\rangle + \alpha^*\beta e^{-i(\omega_2-\omega_1)t}\langle\varphi_1|\hat{D}|\varphi_2\rangle + \alpha\beta^* e^{+i(\omega_2-\omega_1)t}\langle\varphi_2|\hat{D}|\varphi_1\rangle.$$

But the factors $\langle\varphi_1|\hat{D}|\varphi_2\rangle$ and $\langle\varphi_2|\hat{D}|\varphi_1\rangle$ are zero since $\hat{H}$ and $\hat{D}$ commute

So, $\langle\hat{D}\rangle$ never depends on time.

b) Calculate $\langle\hat{A}(t)\rangle$ for $|\psi_0\rangle$ at $t=0$

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$\omega_1 = E_1/\hbar$
$\omega_2 = E_2/\hbar$

$$\langle\hat{A}(t)\rangle = \langle\psi(t)|\hat{A}|\psi(t)\rangle = \langle\psi_0|\hat{U}^+\hat{A}\hat{U}|\psi_0\rangle$$

$$= \frac{1}{2}\left(e^{+i\omega_1 t}\langle\varphi_1| + e^{-i\phi}e^{+i\omega_2 t}\langle\varphi_2|\right)\hat{A}\left(e^{-i\omega_1 t}|\varphi_1\rangle + e^{+i\phi}e^{-i\omega_2 t}|\varphi_2\rangle\right)$$

$$= \frac{1}{2}\left(0 + e^{-i\phi}e^{+i(\omega_2-\omega_1)t}A_0 + e^{+i\phi}e^{-i(\omega_2-\omega_1)t}A_0 + 0\right)$$

$$= \frac{1}{2}\left(2A_0\cos\left((\omega_2-\omega_1)t - \phi\right)\right) = A_0\cos\left((\omega_2-\omega_1)t-\phi\right)$$

Fill in $t = 1$ ns , $\left\{ \begin{array}{l} \omega_2-\omega_1 = \frac{E_2-E_1}{\hbar} = \frac{1.055\cdot 10^{-25}\,\text{J}}{1.055\cdot 10^{-34}\,\text{Js}} \\ \Rightarrow \omega_2-\omega_1 = 1\cdot 10^9\ \text{s}^{-1} \end{array}\right\} \Rightarrow$

$A_0\cos\left((1\cdot 10^9\,\text{s}^{-1})\cdot(1\cdot 10^{-9}\,\text{s}) + \phi\right)$ has a maximum $\Rightarrow$

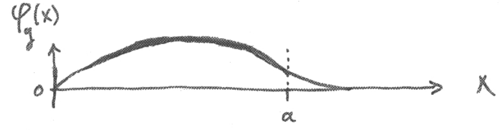
$A_0\cos(1+\phi)$ has a maximum $\Rightarrow$

$1+\phi = 0 \mod 2\pi \Rightarrow \phi = -1 \mod 2\pi \Rightarrow$

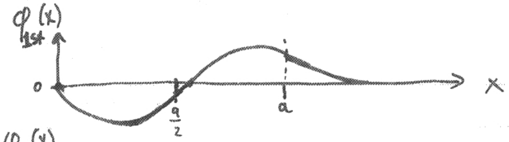
$\boxed{\phi = 2\pi - 1}$ for $0<\phi<2\pi$

Problem 4

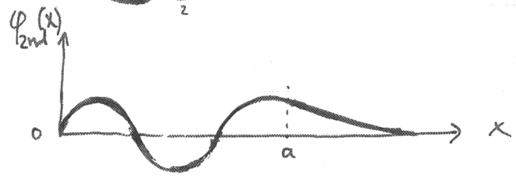
a) Ground state



1^{st} excited state



2^{nd} excited state



b) The ground state energy is lower than that of an infinite potential well of width a, because $\varphi_g(x)$ can penetrate somewhat the region $0<x<b$. Thereby Δx gets larger, and Δp can become lower since for the ground state $\Delta x\,\Delta p \approx \frac{\hbar}{2}$. With lower Δp, the state can have less kinetic energy, and thereby less total energy for the ground state.

c) $\hat{H}\varphi(x) = E\varphi(x)$ with $\hat{H} = \frac{p^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \Rightarrow$

Solve $\frac{\partial^2 \varphi(x)}{\partial x^2} = \frac{2m(E-V)}{\hbar^2}\varphi(x)$

I General solution for $E>V$ (applies to region I) $\quad \varphi_I(x) = Ae^{-ik_1x} + Be^{+ik_1x}$

with here $k_1 = \frac{\sqrt{2mE}}{\hbar}$ since $V(x) = 0$ for $-a<x<0$

II General solution for $V>E$ (applies to region II) $\quad \varphi_{II}(x) = Ce^{-k_2x} + De^{+k_2x}$

with here $k_2 = \frac{\sqrt{2m(V_1-E)}}{\hbar}$

since $V(x) = V_1$ for $0<x<b$

d) Boundary condition at $x=-a$ where $V(x)$ jumps to ∞ $\Rightarrow \varphi_I(-a) = 0 \Rightarrow$

$Ae^{-ik_1a} + Be^{+ik_1a} = 0 \Rightarrow$ express B as $B = -Ae^{-2ik_1a}$

Boundary condition at $x=b$ where $V(x)$ jumps to ∞ $\Rightarrow \varphi_{II}(b) = 0 \Rightarrow$

$Ce^{-k_2b} + De^{+k_2b} = 0 \Rightarrow$ express D as $D = -Ce^{-2k_2b}$

e) Boundary conditions at $x=0$ give

$$\underbrace{\varphi_I(0) = \varphi_{II}(0)}_{e1} \quad \text{and} \quad \underbrace{\left.\frac{d\varphi_I(x)}{dx}\right|_{x=0} = \left.\frac{d\varphi_{II}(x)}{dx}\right|_{x=0}}_{e2}$$

e1: With the result of d) this gives

$$\left.\begin{aligned}\varphi_I(x) &= A\left(e^{-ik_1x} - e^{+ik_1(x-2a)}\right)\\ \varphi_{II}(x) &= C\left(e^{-k_2x} - e^{+k_2(x-2b)}\right)\end{aligned}\right\} \text{at } x=0 \Rightarrow \varphi_I(x) = \varphi_{II}(x)$$

$$A\left(1-e^{-2ik_1a}\right) = C\left(1-e^{-2k_2b}\right) \Rightarrow$$

$$\frac{A}{C} = \frac{1-e^{-2k_2b}}{1-e^{-2ik_1a}}$$

e2:
$$\left.\begin{aligned}\frac{d\varphi_I(x)}{dx} &= -ik_1A\left(e^{-ik_1x} - e^{+ik_1(x-2a)}\right)\\ \frac{d\varphi_{II}(x)}{dx} &= -k_2C\left(e^{-k_2x} - e^{+k_2(x-2b)}\right)\end{aligned}\right\} \text{at } x=0 \; \frac{\partial\varphi_I}{\partial x} = \frac{\partial\varphi_{II}}{\partial x} \Rightarrow$$

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$$-ik_1 A\left(1+e^{-2ik_1 a}\right) = -k_2 C\left(1+e^{-2k_2 b}\right) \Rightarrow$$

$$\frac{C}{A} = \frac{ik_1}{k_2}\,\frac{1+e^{-2ik_1 a}}{1+e^{-2k_2 b}}$$

Use $\frac{A}{C} \cdot \frac{C}{A} = 1$ from $\underline{\underline{e1}}$ and $\underline{\underline{e2}}$ equations $\Rightarrow$

$$\boxed{1 = \frac{ik_1}{k_2} \cdot \frac{1+e^{-2ik_1 a}}{1+e^{-2k_2 b}} \cdot \frac{1-e^{-2k_2 b}}{1-e^{-2ik_1 a}}}$$

Gives the desired equation when noting that $k_1 = \frac{\sqrt{2mE}}{\hbar}$ and $k_2 = \frac{\sqrt{2m(V_1-E)}}{\hbar}$.

(This can be solved reasonbly easy when noting that this can be written as $\tanh(k_2 b) = -\frac{k_2}{k_1}\tan(k_1 a)$)